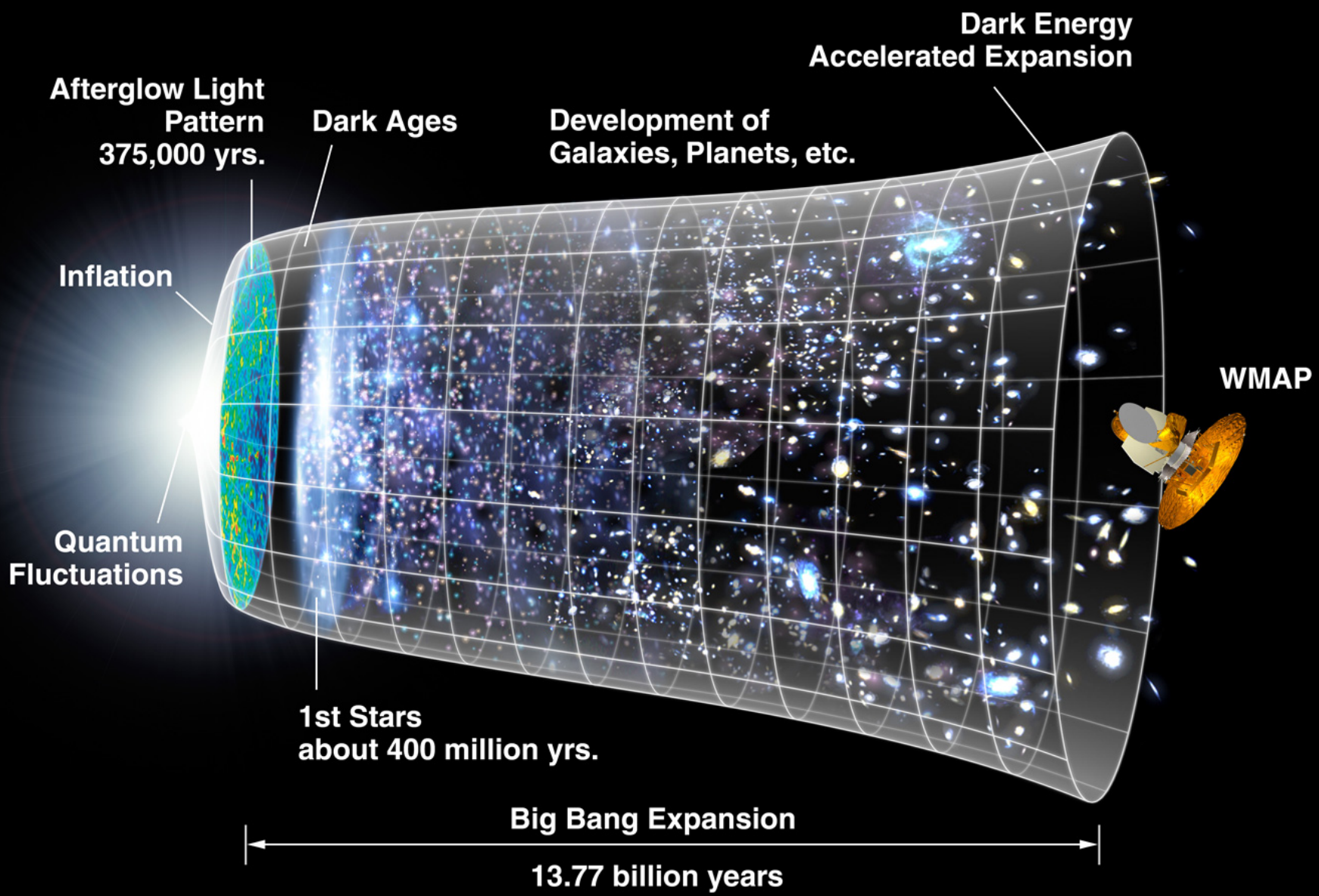


School Observational Cosmology
Angra - Terceira - Açores
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Outline

Lecture 1

- Shortcomings of the Hot Big Bang
- The Inflationary Paradigm
- Homogeneous Scalar Field Dynamics
- Slow roll approximation
- Quantum Fluctuations in de Sitter

Shortcomings of the Hot Big Bang

- **The space-time structure of the observable Universe**
 - Why is the Universe so close to spatial flatness?
 - Why is matter so homogeneously distributed on large scales?
- **The origin of structures in the Universe**
 - How did primordial spectrum of density perturbations originate?
- **The origin of matter and radiation**
 - Where does all the energy in the Universe come from?
 - How did the matter-antimatter asymmetry arise?
- **The initial singularity**
 - Did the Universe have a beginning?
 - What is the global structure of the Universe beyond our observable patch?

Einstein-Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \quad ij + 00$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \quad 00$$

Energy density conservation:

$$D_{\mu}T^{\mu}_{\nu} = 0 \Rightarrow \dot{\rho}(t) + 3\frac{\dot{a}}{a}(\rho(t) + p(t)) = 0$$

$$p(t) = w \rho(t) \quad \text{barotropic fluid}$$

Time evolution of density params

$$x \equiv \Omega_M(a) = \frac{8\pi G \rho_M(a)}{3H^2(a)} = \frac{\Omega_M}{\Omega_M + \Omega_K a + \Omega_\Lambda a^3}$$

$$y \equiv \Omega_\Lambda(a) = \frac{\Lambda}{3H^2(a)} = \frac{\Omega_\Lambda a^3}{\Omega_M + \Omega_K a + \Omega_\Lambda a^3}$$

Homogeneous system of eqs.

$$x' \equiv \frac{dx}{dN} = -x(1 - x + 2y)$$

$$y' \equiv \frac{dy}{dN} = +y(x + 2(1 - y))$$

$$N = \ln a$$

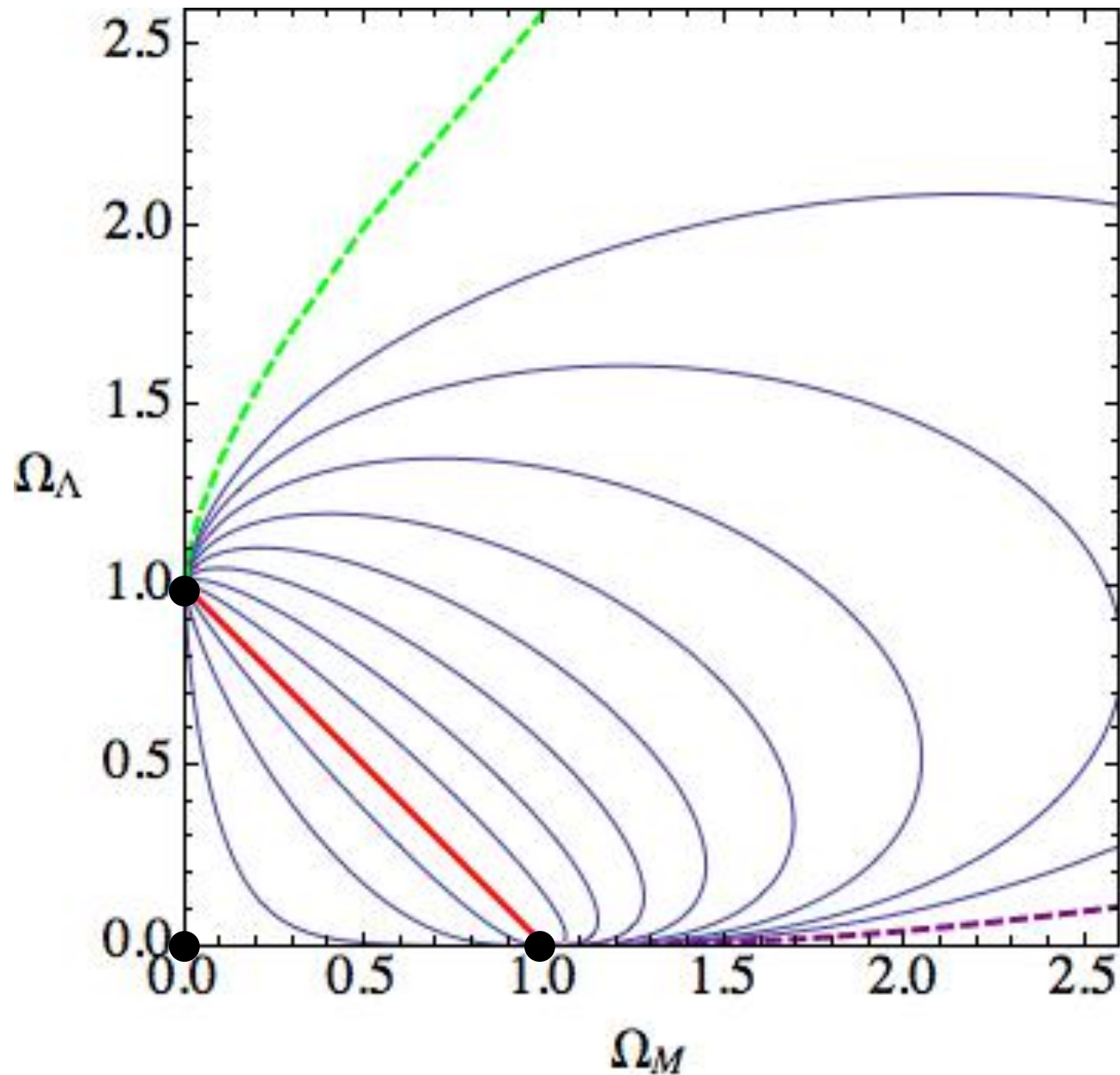
critical points

$$(x = 0, y = 0)$$

$$(x = 1, y = 0)$$

$$(x = 0, y = 1)$$

Flatness is unstable



critical
points

$$(x = 0, y = 0)$$

$$(x = 1, y = 0)$$

$$(x = 0, y = 1)$$

$$\Omega_K = 0,$$

$$\Omega_M + \Omega_\Lambda = 1$$

Flatness

$$H^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2}, \quad \Omega = \frac{8\pi G}{3H^2} \rho, \quad x \equiv \frac{\Omega - 1}{\Omega} = \frac{3K/8\pi G}{\rho a^2}$$

$$\frac{d \ln \rho}{d \ln a} = \frac{\rho'}{\rho} = -3(1+w) \Rightarrow x' = (1+3w)x$$

Matter & Radiation

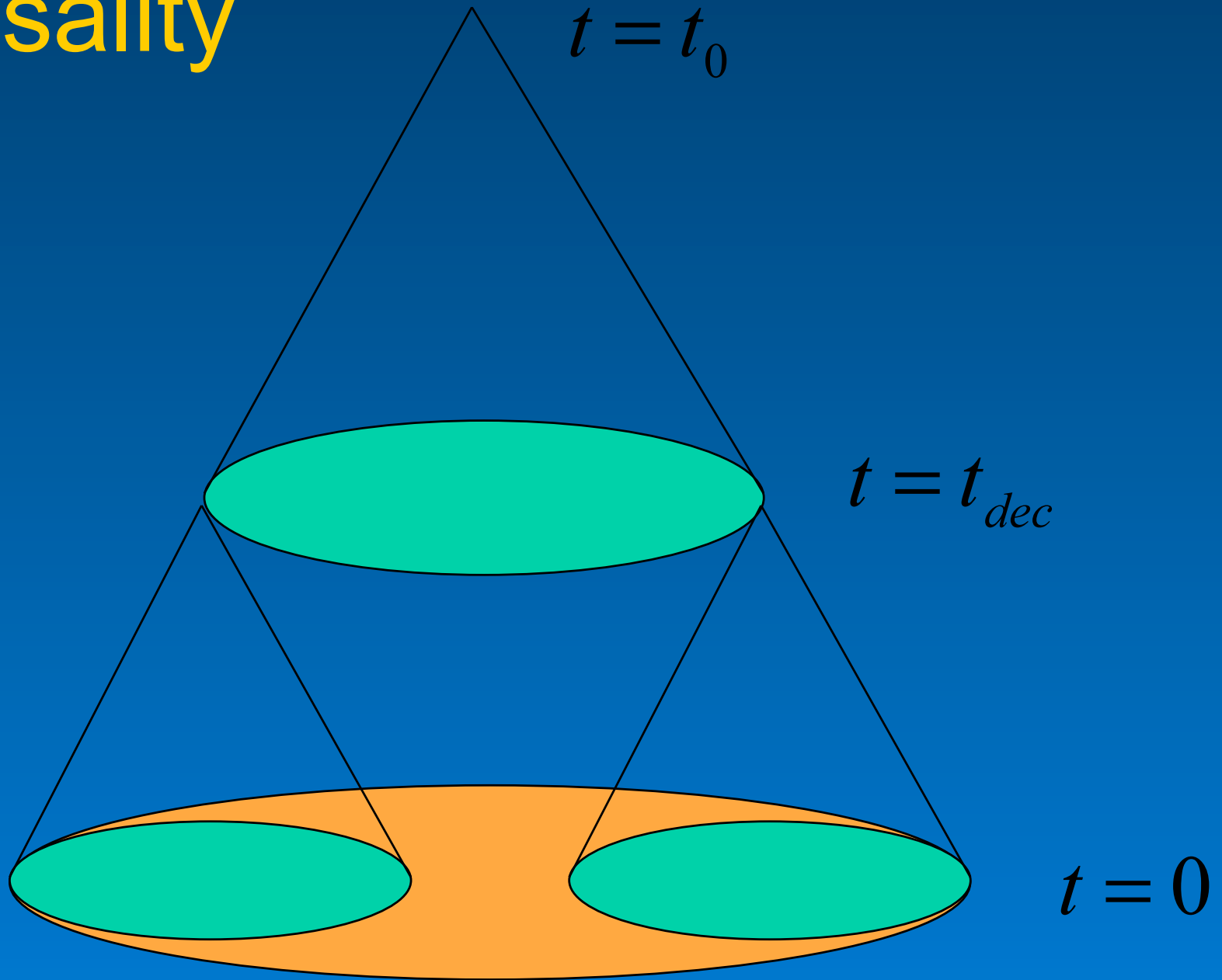
$x = 0$ unstable $x \propto a^2, a$

Vacuum energy

$x = 0$ stable $x \propto a^{-2}$

$$x_0 = x_{in} \left(\frac{T_{in}}{T_{eq}} \right)^2 (1+z_{eq}) < 10^{-3} \quad \text{e.g.} \quad x_{BBN} < 10^{-18}$$

Causality



Homogeneity

Scale Factor

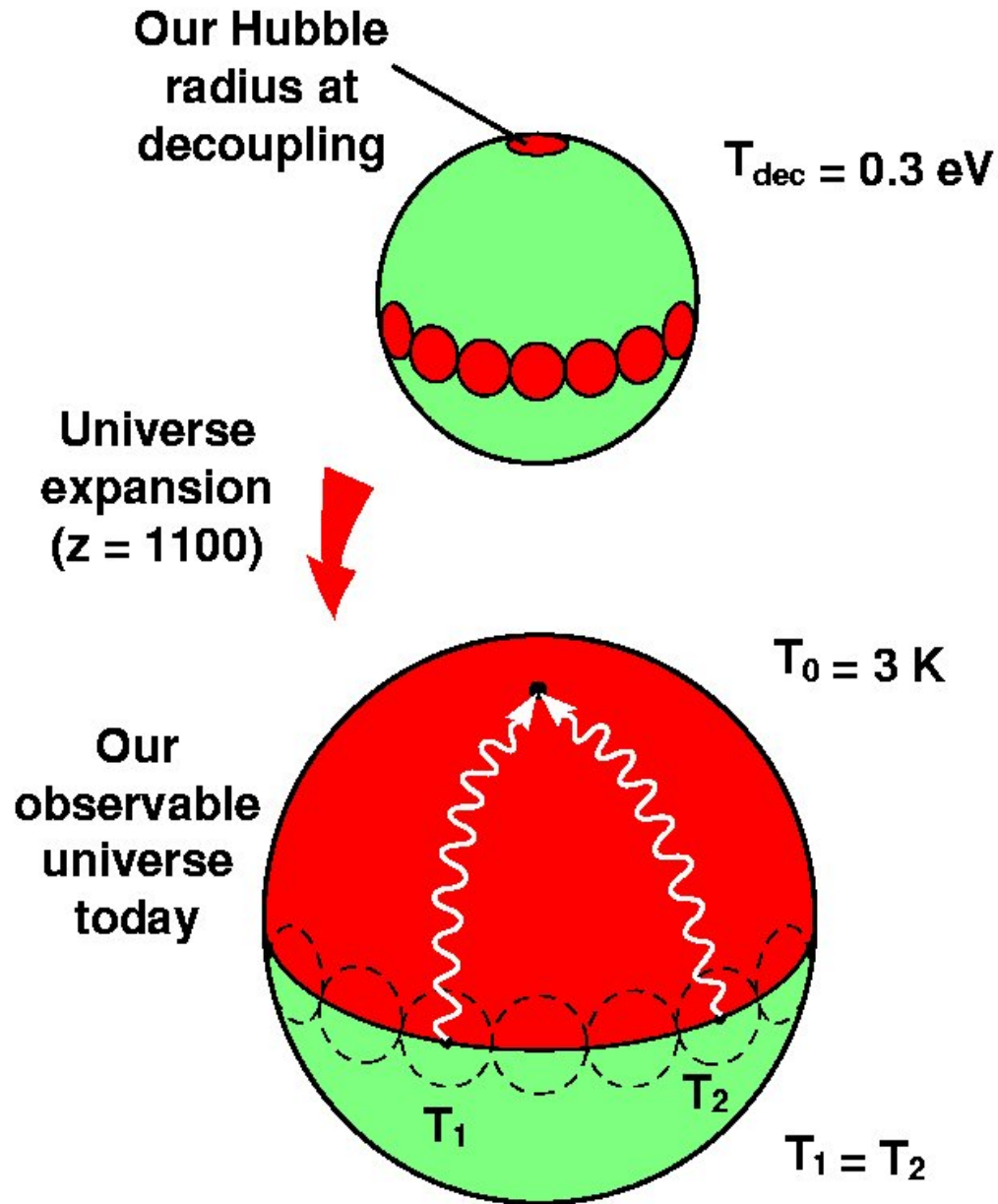
$$a(t) \propto t^p \quad p < 1$$

Particle Horizon

$$d_H(t) = a(t) \int_0^t \frac{dt'}{a(t')} \propto t$$

Causally Disconnected

$$N_{CD}(z) \approx \left(\frac{a}{d_H} \right)^3 \approx (1+z)^{3/2}$$



Homogeneity

Size
Universe

$$a \propto t^{1/2}$$

Horizon

$$d_H \propto t$$

Causally
Disconnected
Regions

$$N_{CD}(z) \approx (1+z)^{3/2}$$

$$a_0 \equiv d_H(t_0)$$

?

Big Bang

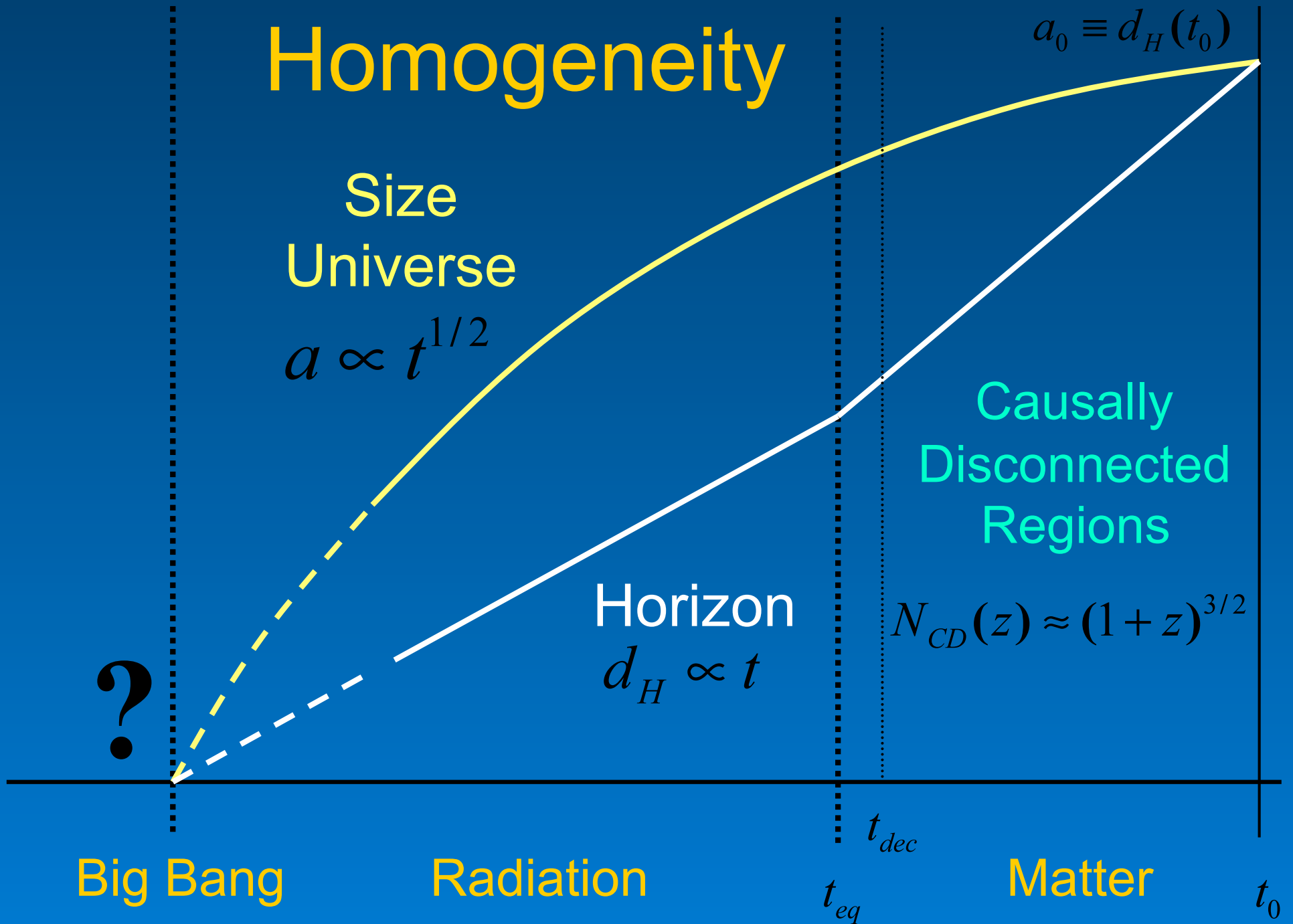
Radiation

Matter

t_{eq}

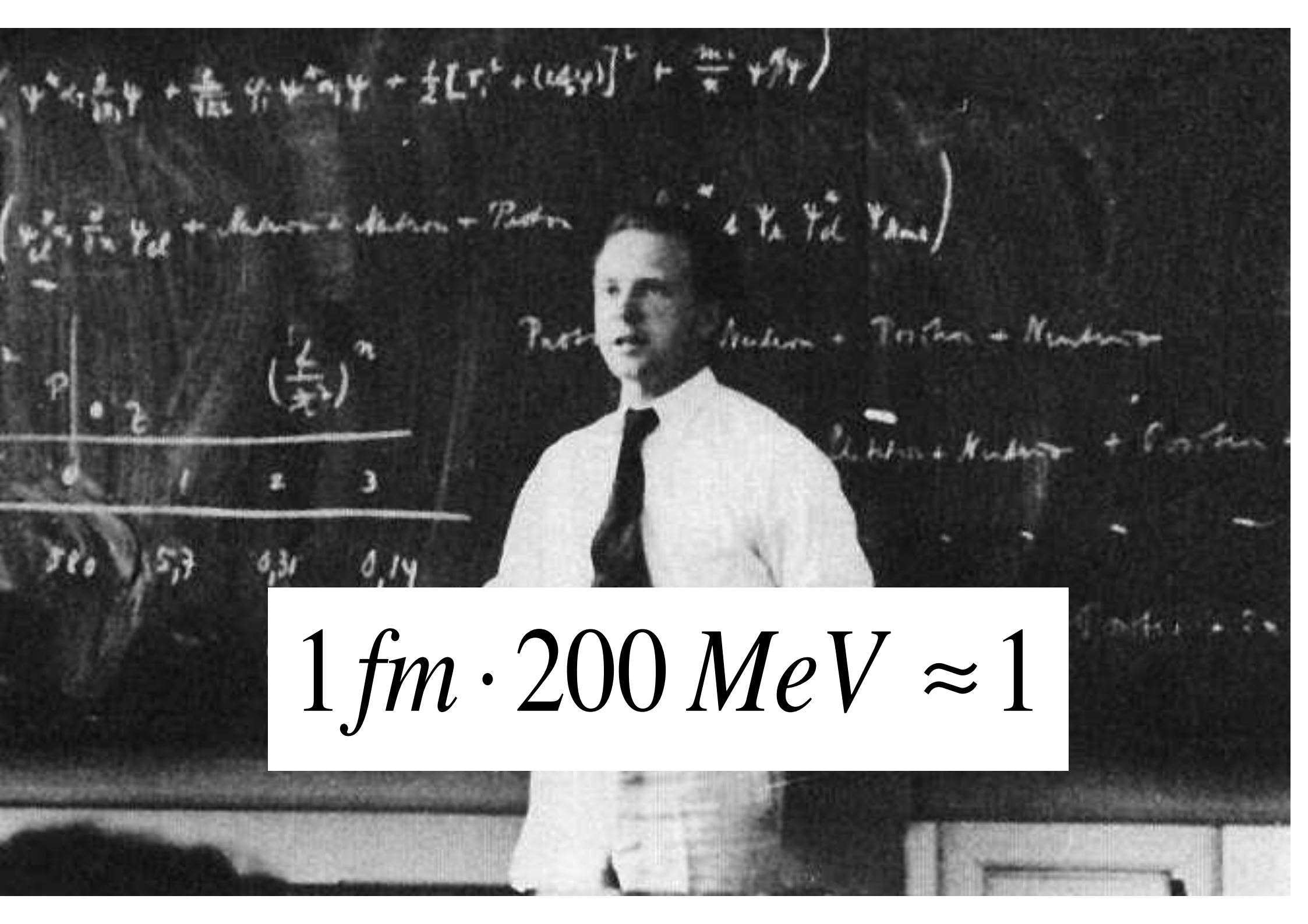
t_{dec}

t_0



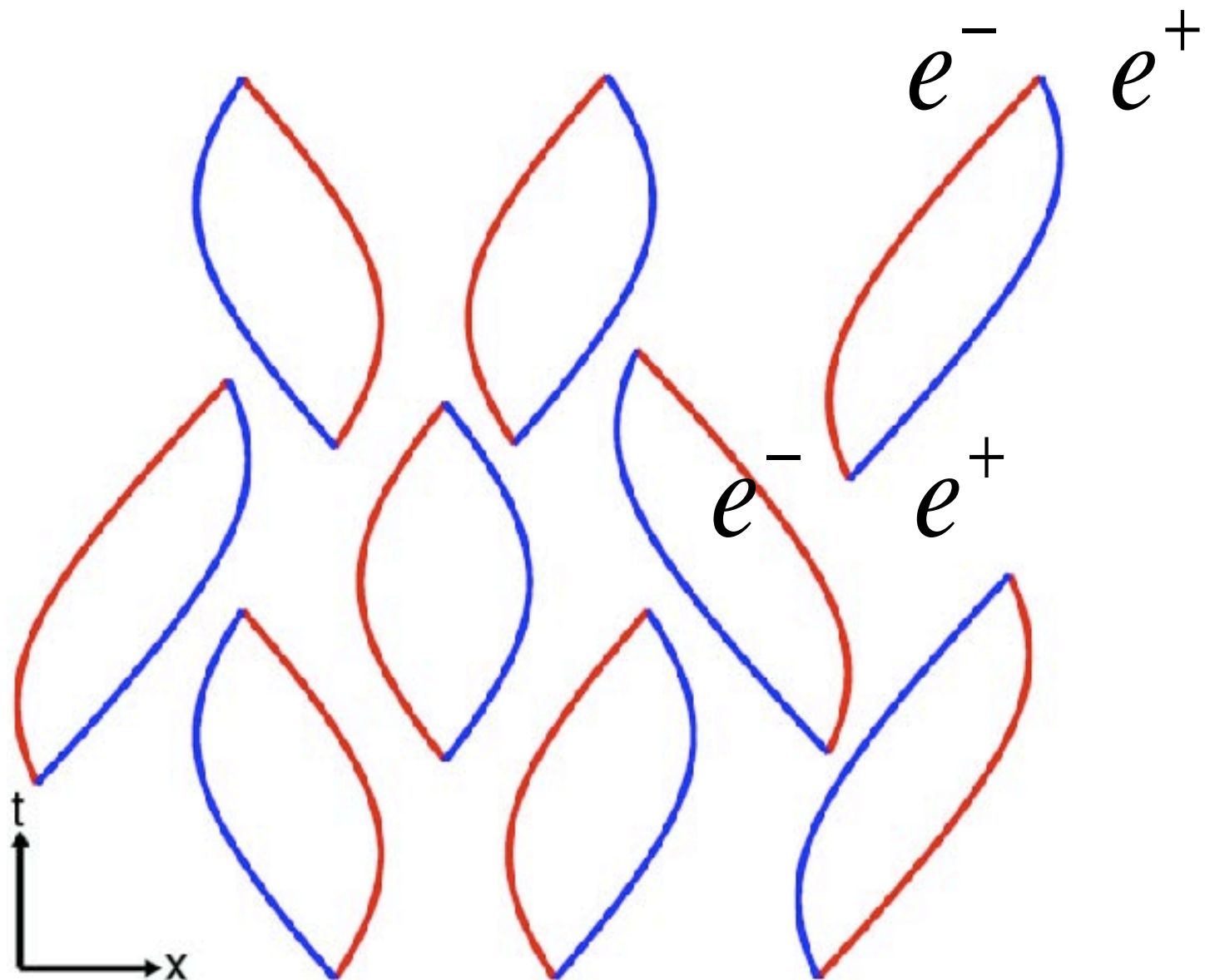
A very elegant solution:

INFLATION



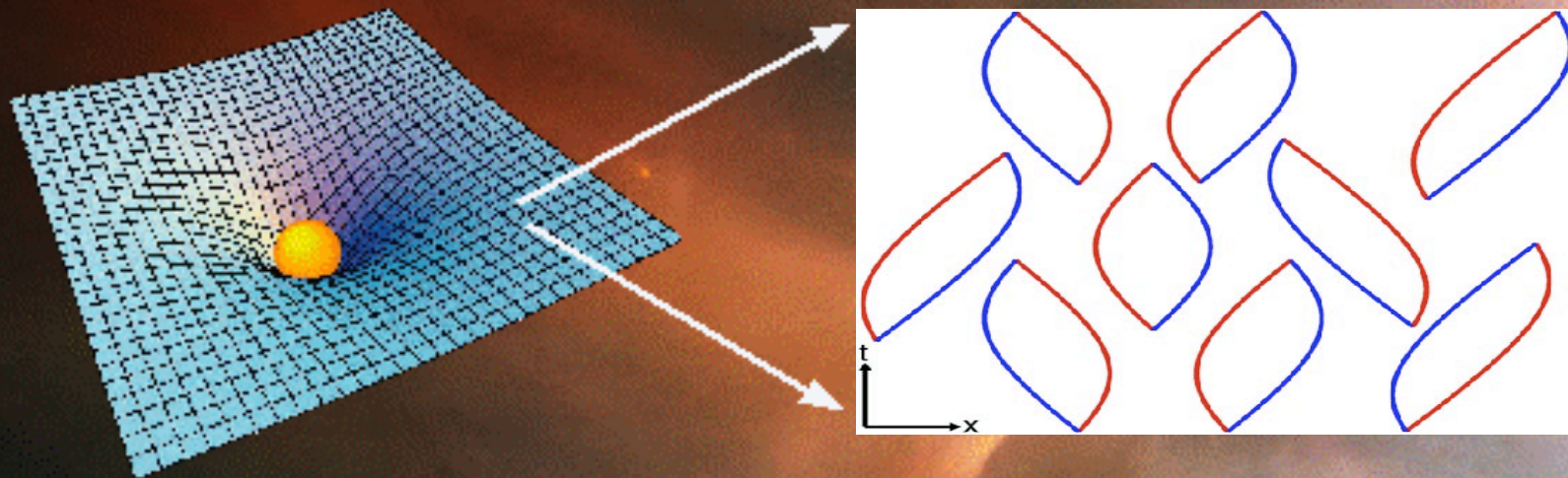
$$1 \text{ fm} \cdot 200 \text{ MeV} \approx 1$$

Vacuum fluctuations



The universe itself could be a product of quantum uncertainty.

“empty space” is a sea of virtual particles winking in and out of existence:



Inflation

Our Universe could be the result of a quantum fluctuation



Alan Guth

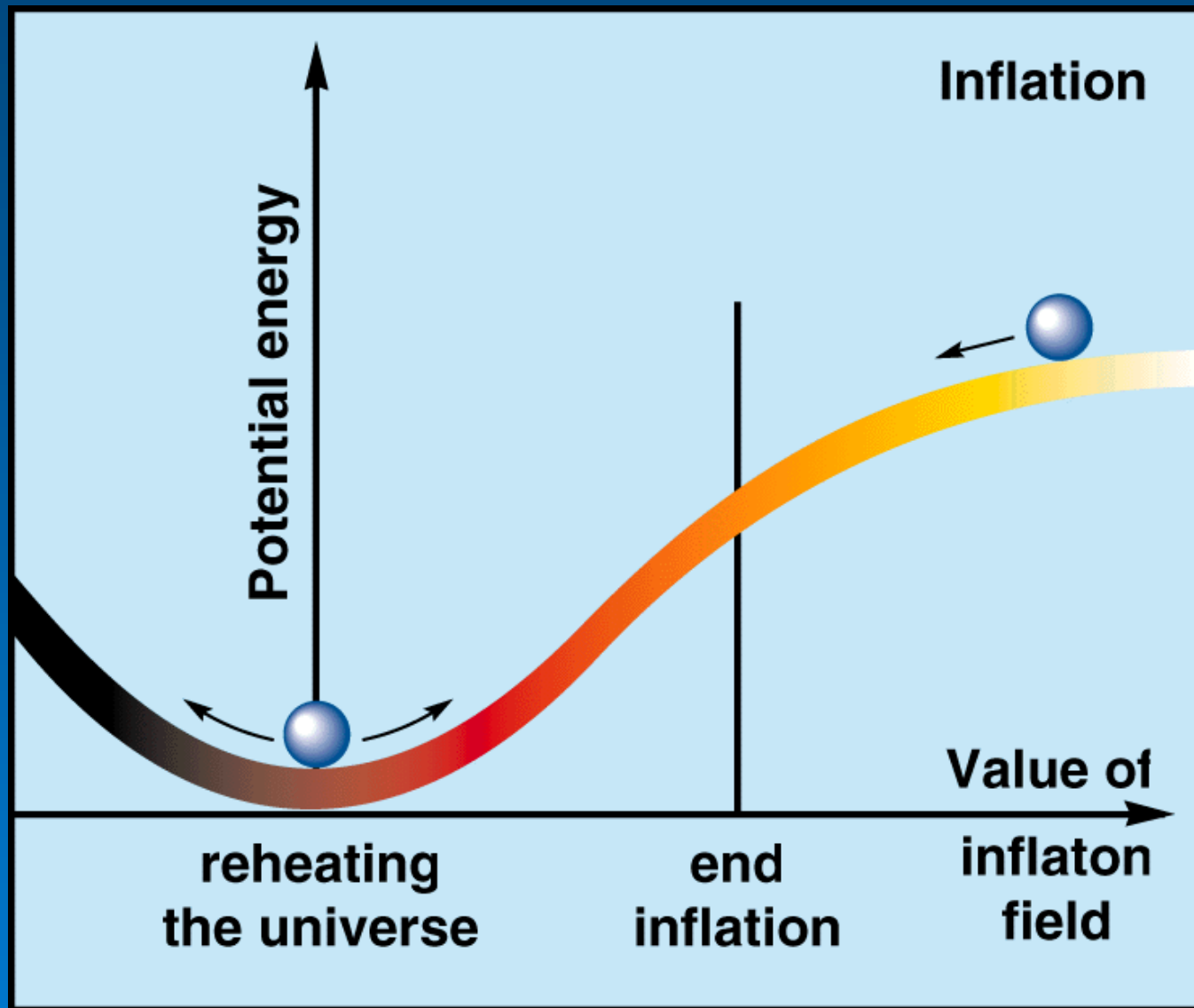


Andrei Linde



A small bubble of quantum vacuum expands very rapidly until it encompasses all our Universe

Effective description (scalar field)



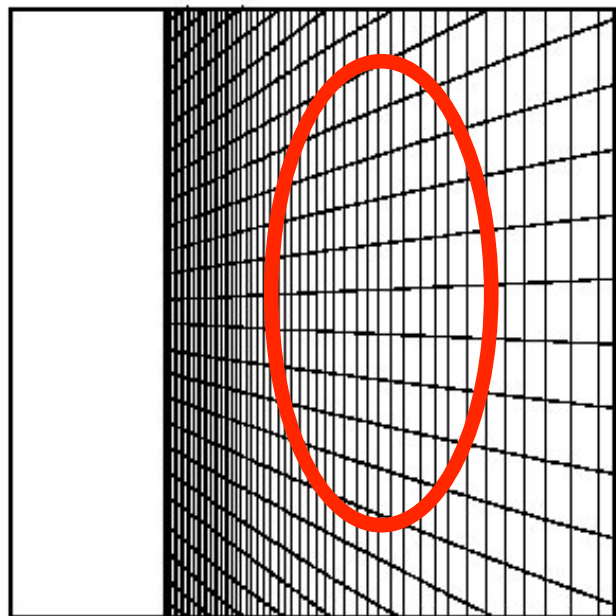
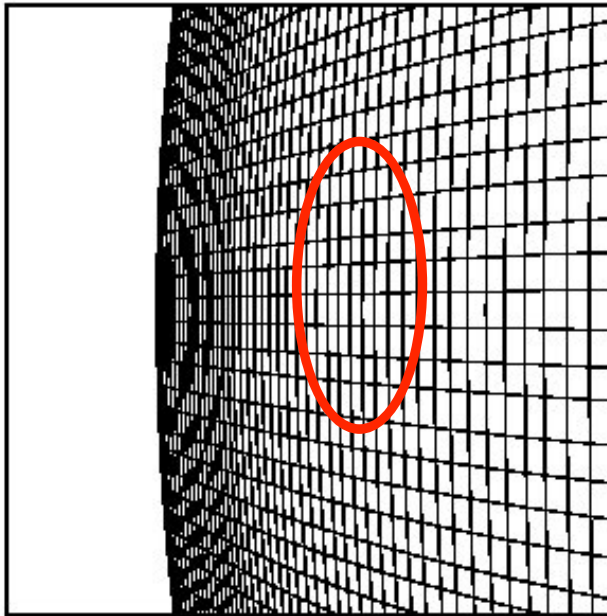
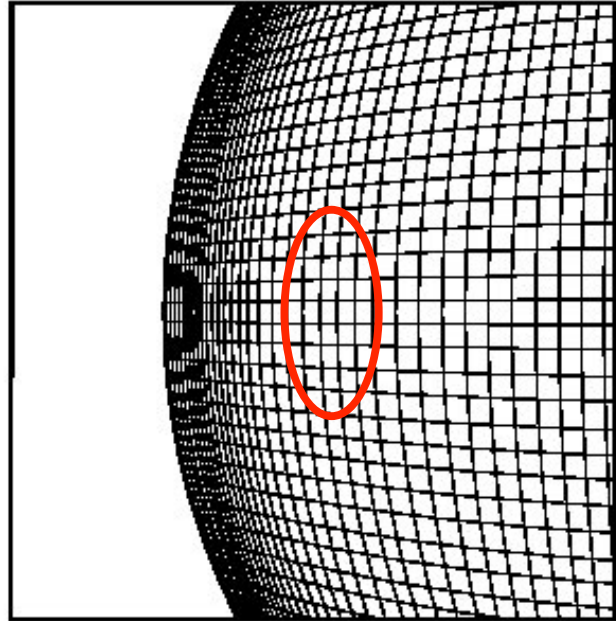
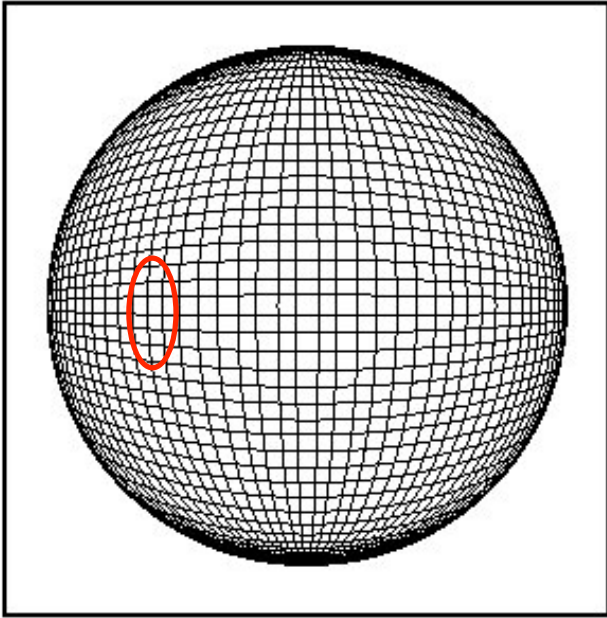
Constant density



Exponential Growth



Flat spatial homogeneous sections



SCALAR FIELD DYNAMICS

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right]$$

Hamiltonian and momentum constraint equations

$$H^2 = \frac{\kappa^2}{3} \left[\frac{1}{2} (\Pi\phi)^2 + V(\phi) \right],$$

$$H_{|i} = -\frac{\kappa^2}{2} \Pi\phi \phi_{|i},$$

Evolution equations

$$\dot{H} = -\frac{\kappa^2}{2} (\Pi\phi)^2,$$

$$\dot{\Pi}\phi + 3H\Pi\phi + \frac{\partial V}{\partial\phi} = 0.$$

HAMILTON-JACOBI EQUATION

Constraint equations $\left(\frac{\partial H}{\partial t}\right)_{\phi} = 0, \quad \left(\frac{\partial \Pi^{\phi}}{\partial t}\right)_{\phi} = 0.$

Then $H \equiv H(\phi(t, x^i)),$

$$3H^2(\phi) = \frac{2}{\kappa^2} \left(\frac{\partial H}{\partial \phi}\right)^2 + \kappa^2 V(\phi),$$

$$\dot{\phi} = -\frac{2}{\kappa^2} \left(\frac{\partial H}{\partial \phi}\right) = \Pi^{\phi}, \quad \frac{\dot{a}}{a} = H(\phi),$$

$$\dot{H} = -\frac{2}{\kappa^2} \left(\frac{\partial H}{\partial \phi}\right)^2 = -\frac{\kappa^2}{2} (\Pi^{\phi})^2,$$

SLOW-ROLL PARAMETERS

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{2}{\kappa^2} \left(\frac{H'(\phi)}{H(\phi)} \right)^2 = -\frac{\partial \ln H}{\partial \ln a},$$

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{2}{\kappa^2} \left(\frac{H''(\phi)}{H(\phi)} \right) = -\frac{\partial \ln H'}{\partial \ln a},$$

The scalar field ϕ acts as a new “time”

$$N_e \equiv \ln \frac{a_{\text{end}}}{a(t)} = \int_t^{t_{\text{end}}} H dt = -\frac{\kappa^2}{2} \int_{\phi}^{\phi_{\text{end}}} \frac{H(\phi) d\phi}{H'(\phi)}$$

The number of “e”-folds N_e to the end inflation

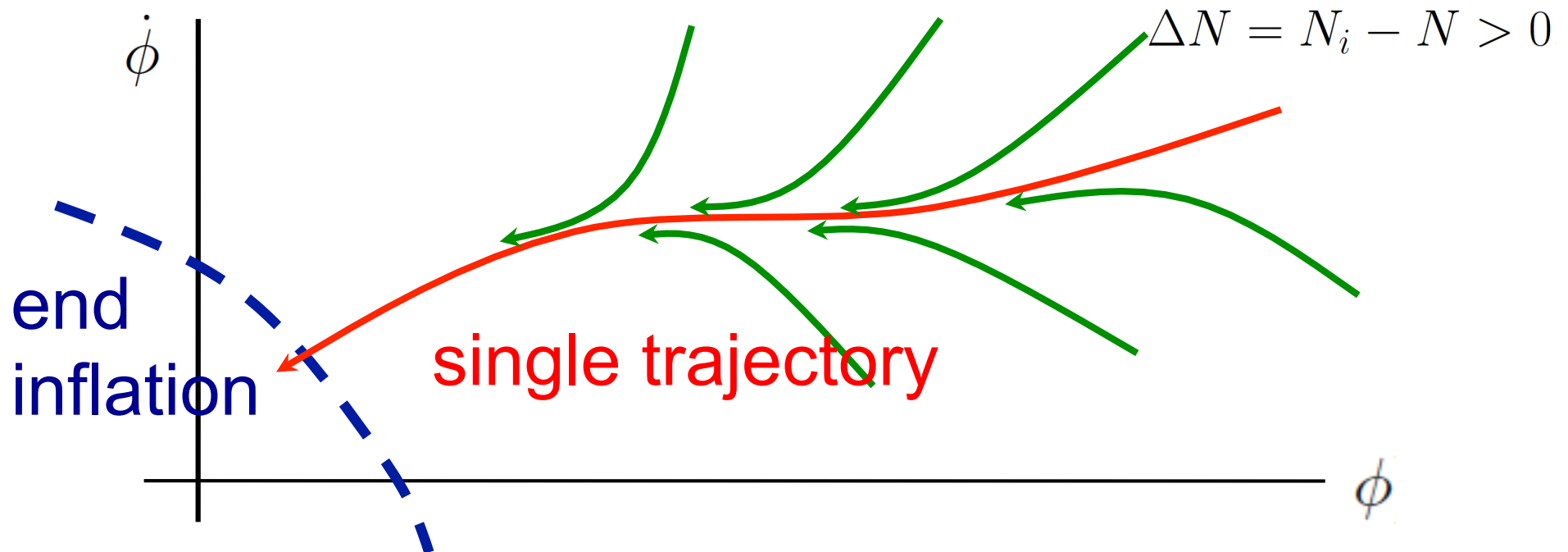
SLOW-ROLL ATTRACTOR

$H_0(\phi)$ exact particular solution

$H(\phi) = H_0(\phi) + \delta H(\phi)$ linear perturbation

Then $H'_0(\phi) \delta H'(\phi) = (3\kappa^2/2) H_0 \delta H$ with solution:

$$\delta H(\phi) = \delta H(\phi_i) \exp\left(\frac{3\kappa^2}{2} \int_{\phi_i}^{\phi} \frac{H_0(\phi) d\phi}{H'_0(\phi)}\right) = \delta H(\phi_i) \exp(-3\Delta N)$$



SLOW-ROLL APPROXIMATION

$$H^2 \left(1 - \frac{\epsilon}{3}\right) \simeq H^2 = \frac{\kappa^2}{3} V(\phi),$$
$$3H\dot{\phi} \left(1 - \frac{\delta}{3}\right) \simeq 3H\dot{\phi} = -V'(\phi)$$

just dynamics

$$\epsilon = \frac{2}{\kappa^2} \left(\frac{H'(\phi)}{H(\phi)} \right)^2 \simeq \frac{1}{2\kappa^2} \left(\frac{V'(\phi)}{V(\phi)} \right)^2 \equiv \epsilon_V \ll 1,$$

$$\delta = \frac{2}{\kappa^2} \frac{H''(\phi)}{H(\phi)} \simeq \frac{1}{\kappa^2} \frac{V''(\phi)}{V(\phi)} - \frac{1}{2\kappa^2} \left(\frac{V'(\phi)}{V(\phi)} \right)^2 \equiv \eta_V - \epsilon_V \ll 1,$$

$$\xi = \frac{4}{\kappa^4} \frac{H'(\phi)H'''(\phi)}{H^2(\phi)} \simeq \frac{1}{\kappa^4} \frac{V'(\phi)V'''(\phi)}{V^2(\phi)} - \frac{3}{2\kappa^4} \frac{V''(\phi)}{V(\phi)} \left(\frac{V'(\phi)}{V(\phi)} \right)^2$$

$$+ \frac{3}{4\kappa^4} \left(\frac{V'(\phi)}{V(\phi)} \right)^4 \equiv \xi_V - 3\eta_V\epsilon_V + 3\epsilon_V^2 \ll 1.$$

$$N \simeq \int_{\phi_i}^{\phi_e} \frac{\kappa d\phi}{\sqrt{2\epsilon_V(\phi)}} = \kappa^2 \int_{\phi_i}^{\phi_e} \frac{V(\phi) d\phi}{V'(\phi)},$$

INFLATIONARY SOLUTIONS OF HBBP

$$\begin{aligned} \rho &= \frac{1}{2} \dot{\phi}^2 + V(\phi), \\ p &= \frac{1}{2} \dot{\phi}^2 - V(\phi). \end{aligned} \quad \dot{\rho} + 3H(\rho + p) = 0$$

$$V(\phi) \gg \dot{\phi}^2 \Rightarrow p \simeq -\rho \Rightarrow \rho \simeq \text{const.} \Rightarrow H(\phi) \simeq \text{const.}$$

$$a(t) \sim \exp(Ht) \Rightarrow \frac{\ddot{a}}{a} > 0 \quad \text{accelerated expansion}$$

$$x_0 = x_{\text{in}} e^{-2N} \frac{a_{\text{rh}}^2 \rho_{\text{rh}}}{a_{\text{end}}^2 \rho_{\text{end}}} \frac{T_{\text{rh}}^2}{T_{\text{eq}}^2} (1 + z_{\text{eq}}) \simeq e^{-2N} 10^{56} \leq 1 \Rightarrow N \geq 65$$

$$\begin{aligned} {}^{(3)}R &= \frac{6K}{a^2} = {}^{(3)}R_{\text{in}} e^{-2N} \longrightarrow 0, & \rho_{\text{M}} &\propto a^{-3} \sim e^{-3N} \longrightarrow 0, \\ \delta_k &\sim \left(\frac{k}{aH}\right)^2 \Phi_k \propto e^{-2N} \longrightarrow 0, & \rho_{\text{R}} &\propto a^{-4} \sim e^{-4N} \longrightarrow 0, \end{aligned}$$

curvature

matter

LINEAR METRIC PERTURBATIONS

$$ds^2 = a(\eta)^2 [-(1 + 2\Phi)d\eta^2 + (1 - 2\Phi)d\mathbf{x}^2 + h_{ij} dx^i dx^j]$$

linear pert. eqs.

$$\Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi = \frac{\kappa^2}{2}[\phi'\delta\phi' - a^2V'(\phi)\delta\phi],$$

$$-\nabla^2\Phi + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi = -\frac{\kappa^2}{2}[\phi'\delta\phi' + a^2V'(\phi)\delta\phi],$$

$$\Phi' + \mathcal{H}\Phi = \frac{\kappa^2}{2}\phi'\delta\phi,$$

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2V''(\phi)\delta\phi = 4\phi'\Phi' - 2a^2V'(\phi)\Phi.$$

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0,$$
$$\nabla^2\Phi = \frac{\kappa^2}{2} \frac{\mathcal{H}}{a^2} (zu' - z'u),$$
$$\left(\frac{a^2\Phi}{\mathcal{H}}\right)' = \frac{\kappa^2}{2} zu.$$

Mukhanov variables

$$u \equiv a\delta\phi + z\Phi,$$
$$z \equiv a \frac{\phi'}{\mathcal{H}}.$$

QUANTUM FLUCTUATIONS IN dS

$$\delta S = \frac{1}{2} \int d^3x d\eta \left[(u')^2 - (\nabla u)^2 + \frac{z''}{z} u^2 \right]$$

$$\hat{u}(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[u_k(\eta) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\eta) \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right],$$

scalar field's Fock space

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}'),$$

$$\hat{a}_{\mathbf{k}} |0\rangle = 0.$$

equal-time commutation relations

$$[\hat{u}(\eta, \mathbf{x}), \hat{\Pi}_u^\dagger(\eta, \mathbf{x}')] = i\hbar \delta^3(\mathbf{x} - \mathbf{x}')$$

QUANTUM FLUCTUATIONS IN dS

normalization condition on the modes u_k

$$u_k u_k^{*'} - u_k' u_k^* = i$$

the Wronskian of the mode equation

$$u_k'' + \left(k^2 - \frac{z''}{z} \right) u_k = 0$$

Schrödinger like equation

$$-u_k'' + U(\eta) u_k = k^2 u_k$$

time-dependent potential

$$U(\eta) = z''/z$$

SOLUTIONS OF MODE EQUATIONS

slow-roll parameters

$$\epsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = \frac{\kappa^2 z^2}{2 a^2},$$

$$\delta = 1 - \frac{\phi''}{\mathcal{H}\phi'} = 1 + \epsilon - \frac{z'}{\mathcal{H}z},$$

$$\xi = - \left(2 - \epsilon - 3\delta + \delta^2 - \frac{\phi'''}{\mathcal{H}^2\phi'} \right)$$

approx. constant

$$\epsilon' = 2\mathcal{H} \left(\epsilon^2 - \epsilon\delta \right) = \mathcal{O}(\epsilon^2),$$

$$\delta' = \mathcal{H} \left(\epsilon\delta - \xi \right) = \mathcal{O}(\epsilon^2).$$

for constant slow-roll parameters, we can write

$$\eta = \frac{-1}{\mathcal{H}} + \int \frac{\epsilon da}{a\mathcal{H}} \simeq \frac{-1}{\mathcal{H}} \frac{1}{1-\epsilon},$$

$$\frac{z''}{z} = \mathcal{H}^2 \left[(1 + \epsilon - \delta)(2 - \delta) + \mathcal{H}^{-1}(\epsilon' - \delta') \right] \simeq \frac{1}{\eta^2} \left(\nu^2 - \frac{1}{4} \right),$$

$$\text{where } \nu = \frac{1 + \epsilon - \delta}{1 - \epsilon} + \frac{1}{2}$$

EXACT SOLUTIONS OF MODE EQS.

two asymptotic regimes,

$$u_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad k \gg aH \quad \text{Minkowsky}$$

$$u_k = C_1(k) z \quad k \ll aH \quad \text{superhorizon}$$

exact solution that connects the two regimes

$$u_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu + \frac{1}{2})\frac{\pi}{2}} (-\eta)^{1/2} H_\nu^{(1)}(-k\eta)$$

where $H_\nu^{(1)}(z)$ is the Hankel function of the first kind

$$\text{e.g. } H_{3/2}^{(1)}(x) = -e^{ix} \sqrt{2/\pi x} (1 + i/x),$$

$$\text{and } \nu \text{ is given by } \nu = \frac{1 + \epsilon - \delta}{1 - \epsilon} + \frac{1}{2}$$

EXACT SOLUTIONS OF MODE EQS.

limit $k\eta \rightarrow 0$, the solution becomes

$$|u_k| = \frac{2^{\nu-\frac{3}{2}} \Gamma(\nu)}{\sqrt{2k} \Gamma(\frac{3}{2})} (-k\eta)^{\frac{1}{2}-\nu} \equiv \frac{C(\nu)}{\sqrt{2k}} \left(\frac{k}{aH}\right)^{\frac{1}{2}-\nu},$$

$$C(\nu) = 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} (1-\epsilon)^{\nu-\frac{1}{2}} \simeq 1 \quad \text{for } \epsilon, \delta \ll 1$$

compute Φ and $\delta\phi$ from the super-Hubble-scale mode

$$\Phi = C_1 \left(1 - \frac{\mathcal{H}}{a^2} \int a^2 d\eta\right) + C_2 \frac{\mathcal{H}}{a^2},$$

$$\frac{\delta\phi}{\phi'} = \frac{C_1}{a^2} \int a^2 d\eta - \frac{C_2}{a^2}. \quad \begin{array}{l} C_1 \text{ growing mode} \\ C_2 \text{ decaying mode} \end{array}$$

SCALAR CURVATURE PERTURBATIONS

gauge invariant quantity *constant* for superhorizon modes of adiabatic perturbations,

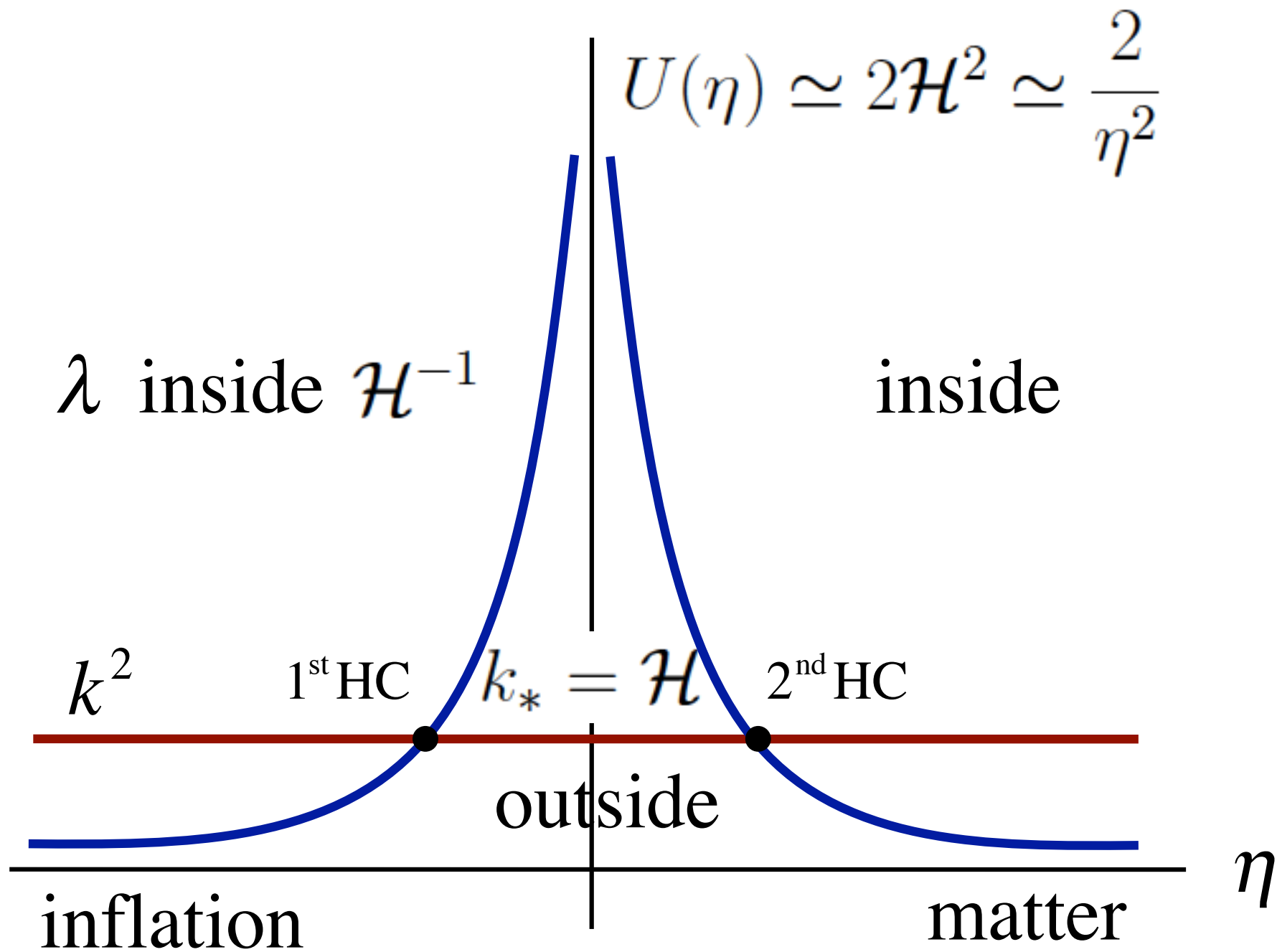
$$\zeta \equiv \Phi + \frac{1}{\epsilon\mathcal{H}} (\Phi' + \mathcal{H}\Phi) = \frac{u}{z},$$

ζ is the gauge invariant curvature perturbation \mathcal{R}_c on constant energy density hypersurfaces,

$$\zeta = \mathcal{R}_c + \frac{1}{\epsilon\mathcal{H}^2} \nabla^2\Phi$$

$$\zeta' = \frac{1}{\epsilon\mathcal{H}} \nabla^2\Phi \simeq 0 \quad \text{constant}$$

for (adiabatic) superhorizon modes, $k \ll aH$



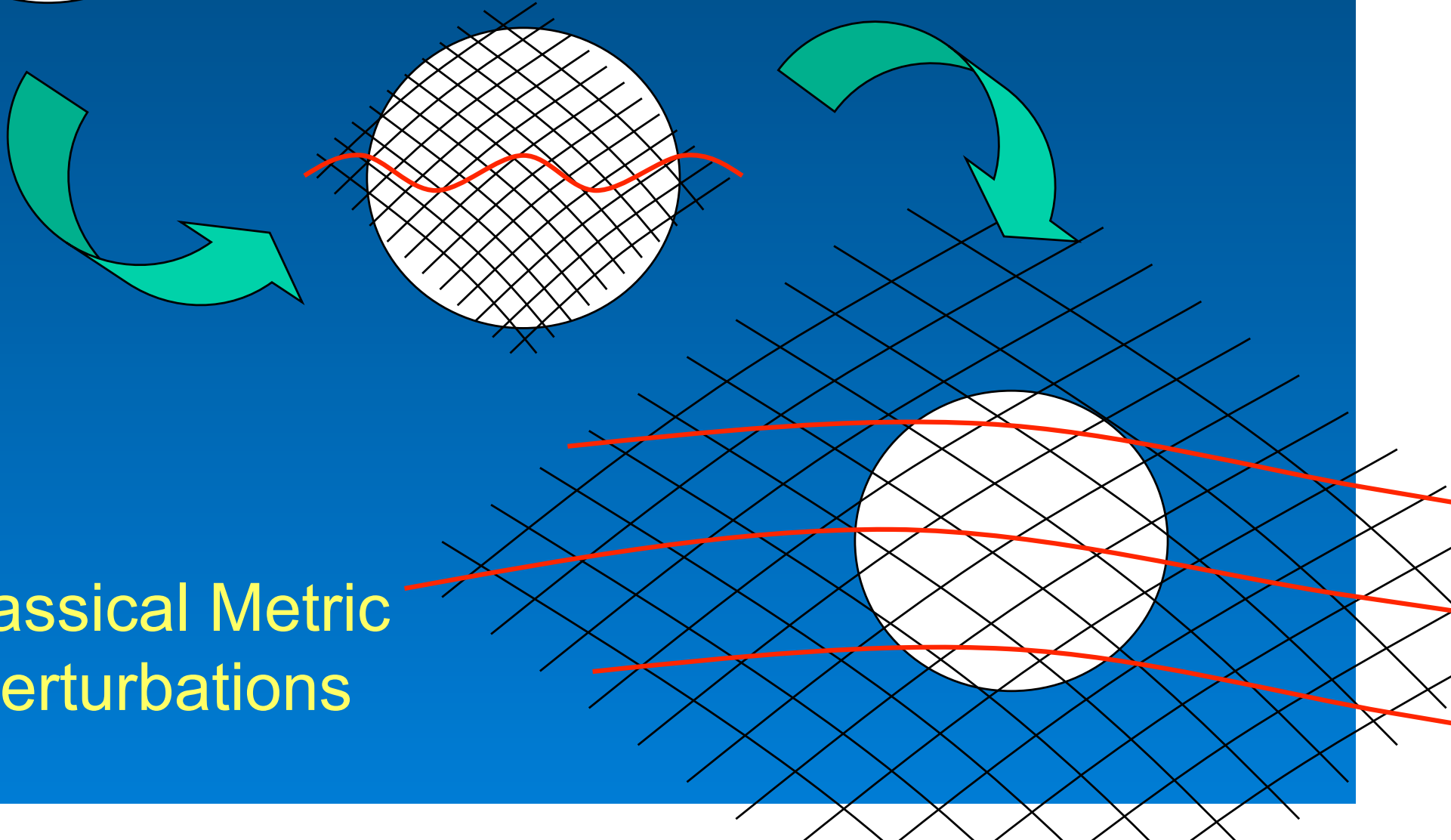
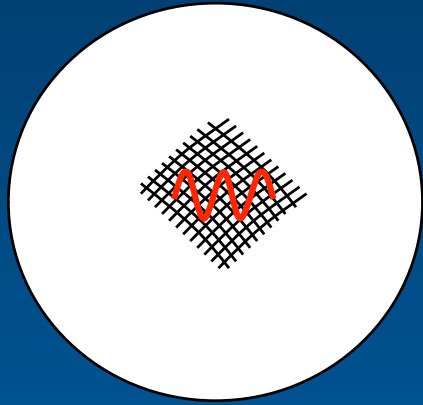
SCALAR CURVATURE PERTURBATIONS

Therefore, we can evaluate the Newtonian potential Φ_k when the perturbation reenters the horizon during radiation/matter eras in terms of the curvature perturbation \mathcal{R}_k when it left the Hubble scale during inflation,

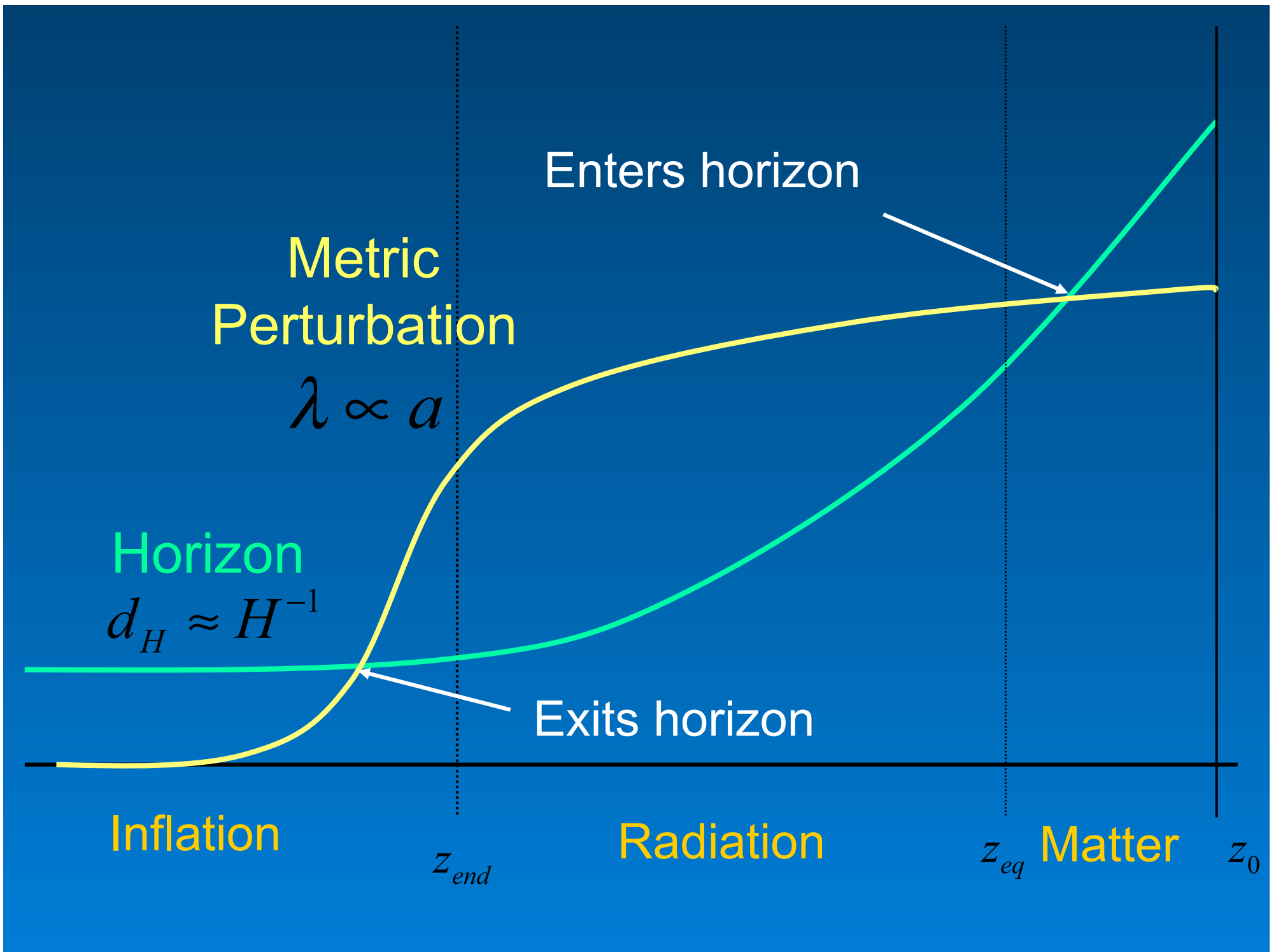
$$\Phi_k = \left(1 - \frac{\mathcal{H}}{a^2} \int a^2 d\eta\right) \mathcal{R}_k = \frac{3 + 3\omega}{5 + 3\omega} \mathcal{R}_k = \begin{cases} \frac{2}{3} \mathcal{R}_k & \text{radiation era,} \\ \frac{3}{5} \mathcal{R}_k & \text{matter era.} \end{cases}$$

These expressions will be of special importance later.

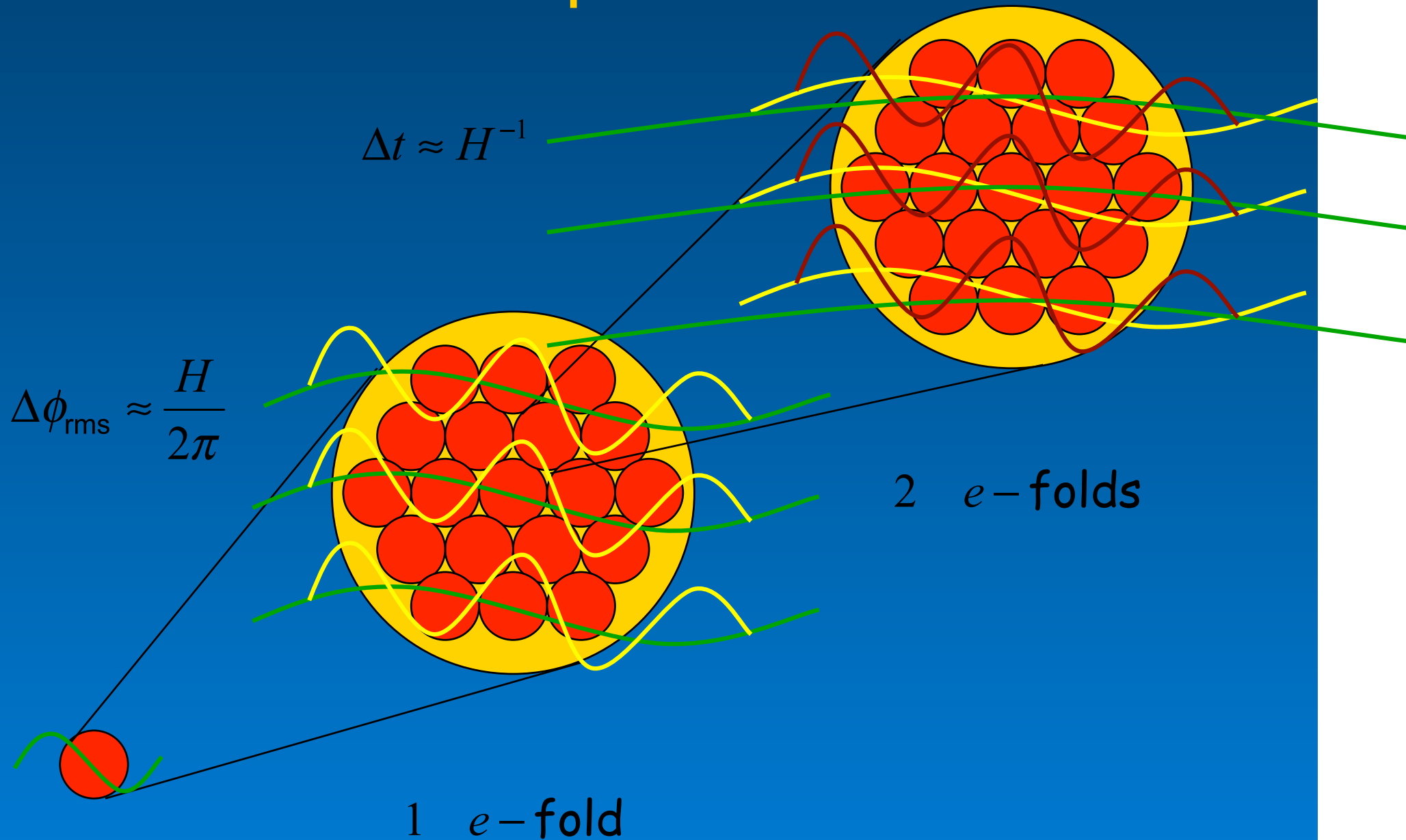
Quantum Fluctuations within the horizon

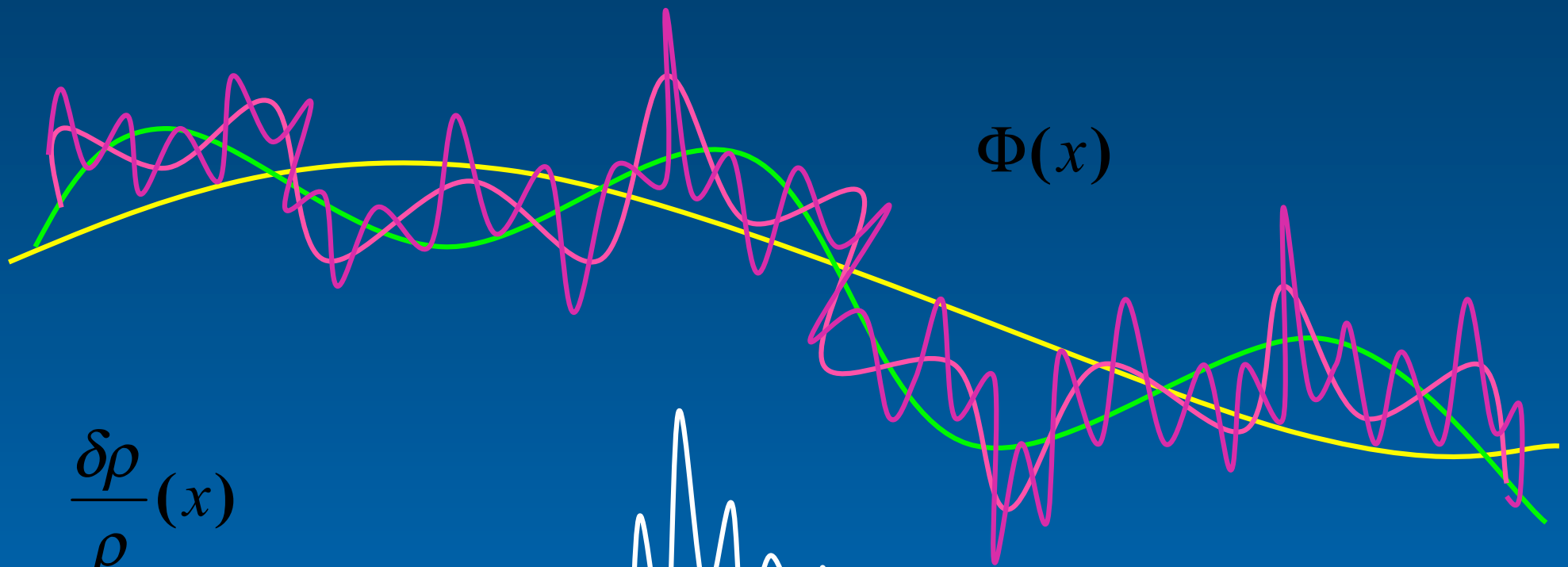


Classical Metric
perturbations

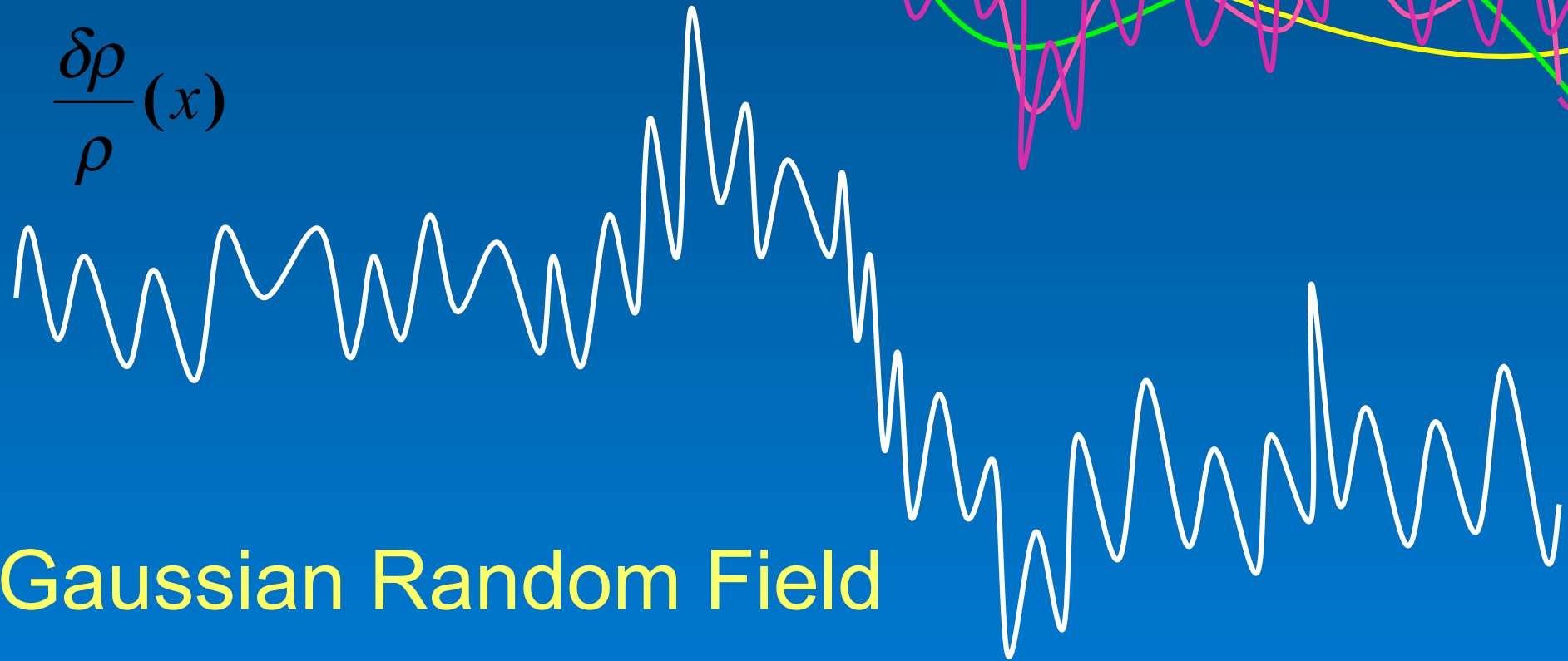


Scale Invariant Spectrum





$$\frac{\delta\rho}{\rho}(x)$$



Gaussian Random Field

Horizon Crossing

perturbation

horizon

causal region

Inflation

Radiation

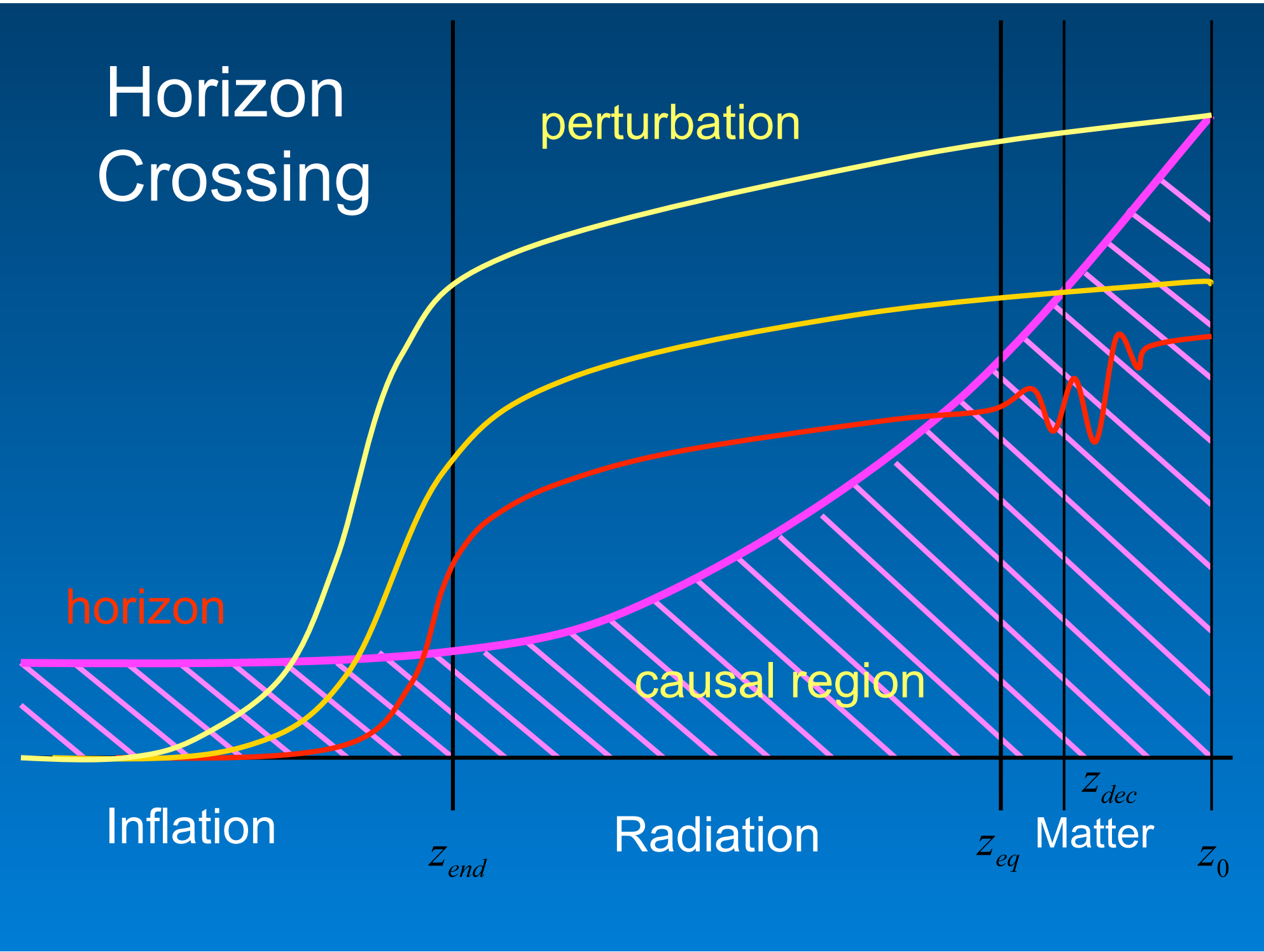
Matter

z_{end}

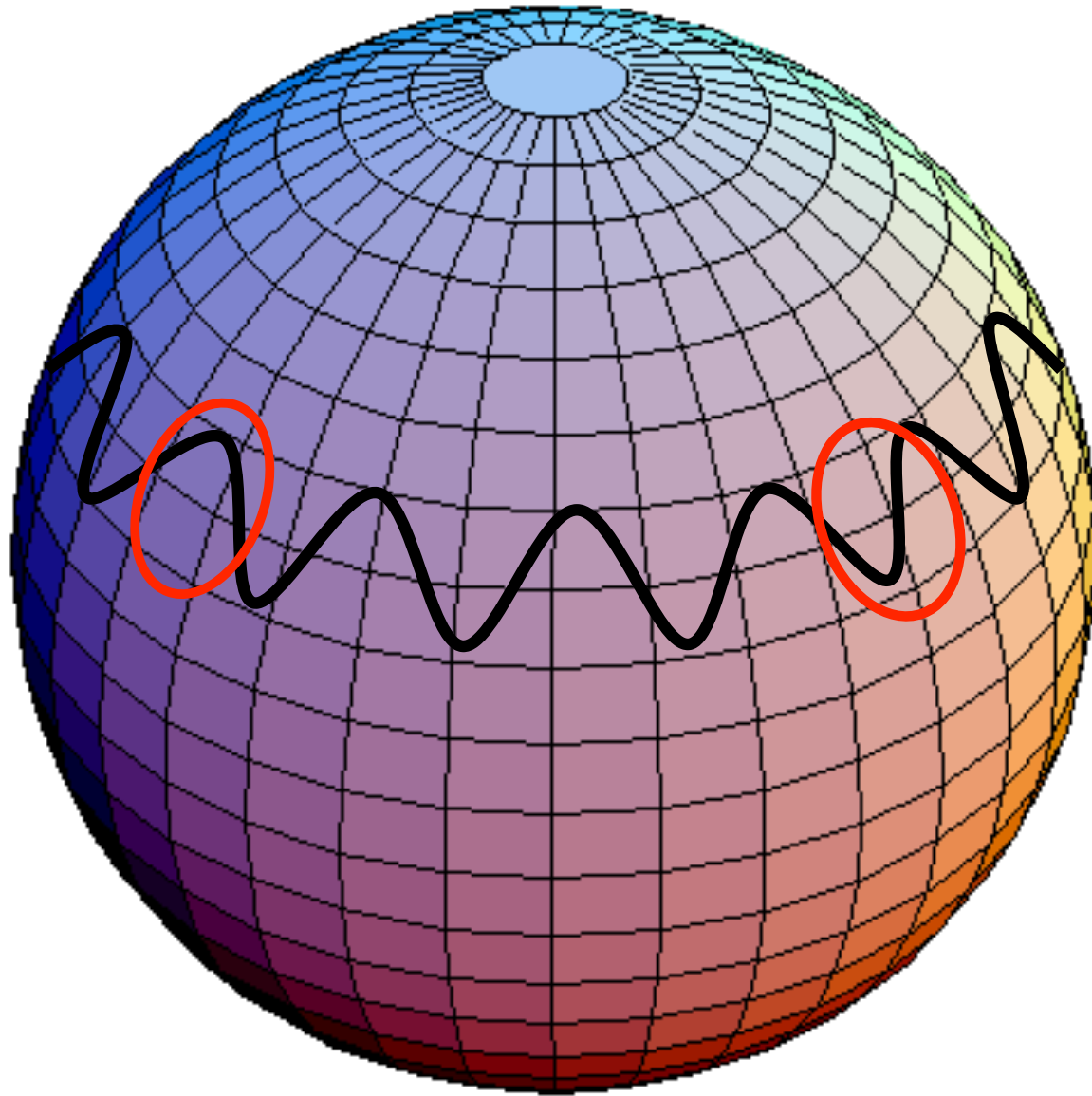
z_{eq}

z_{dec}

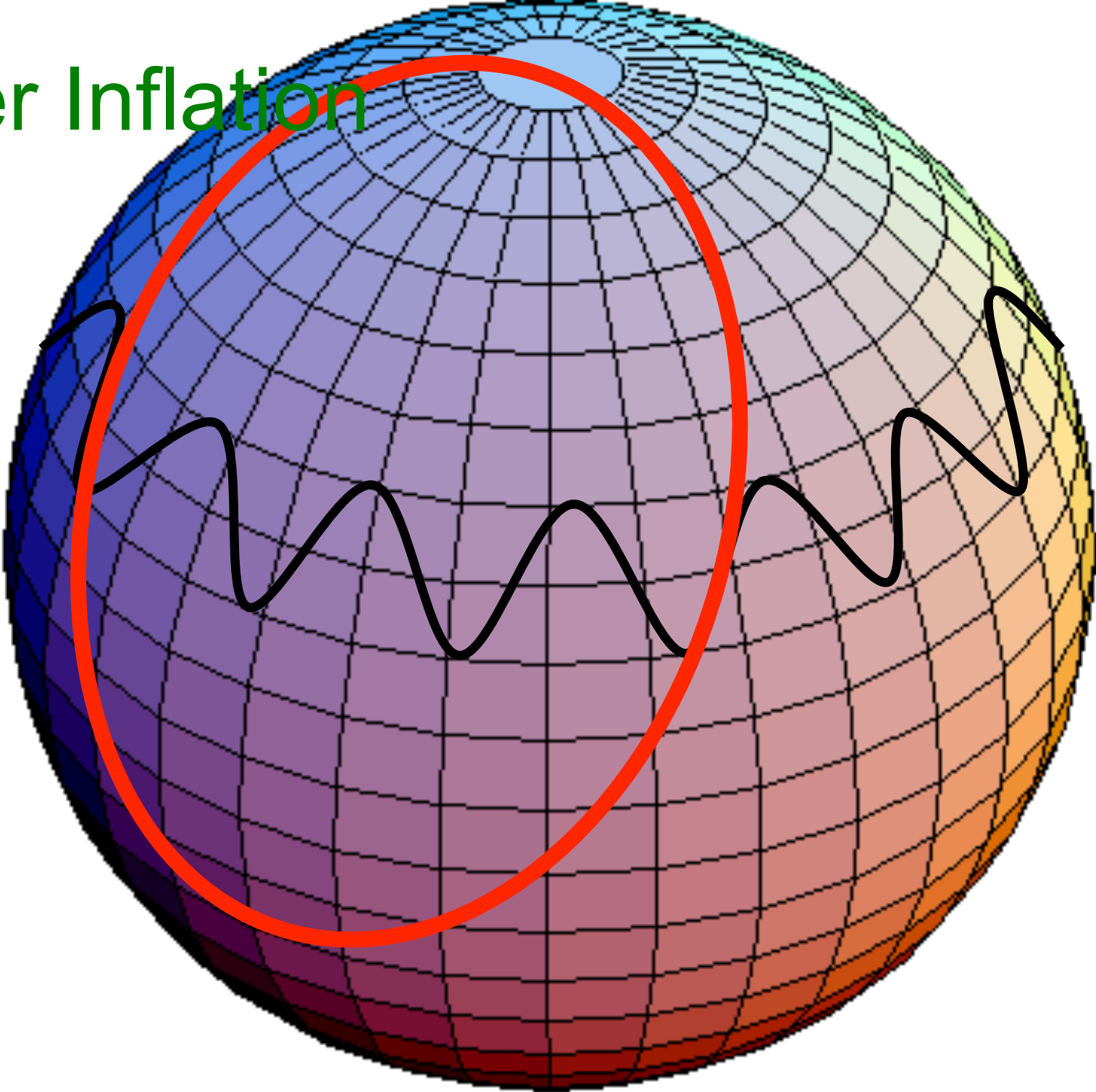
z_0



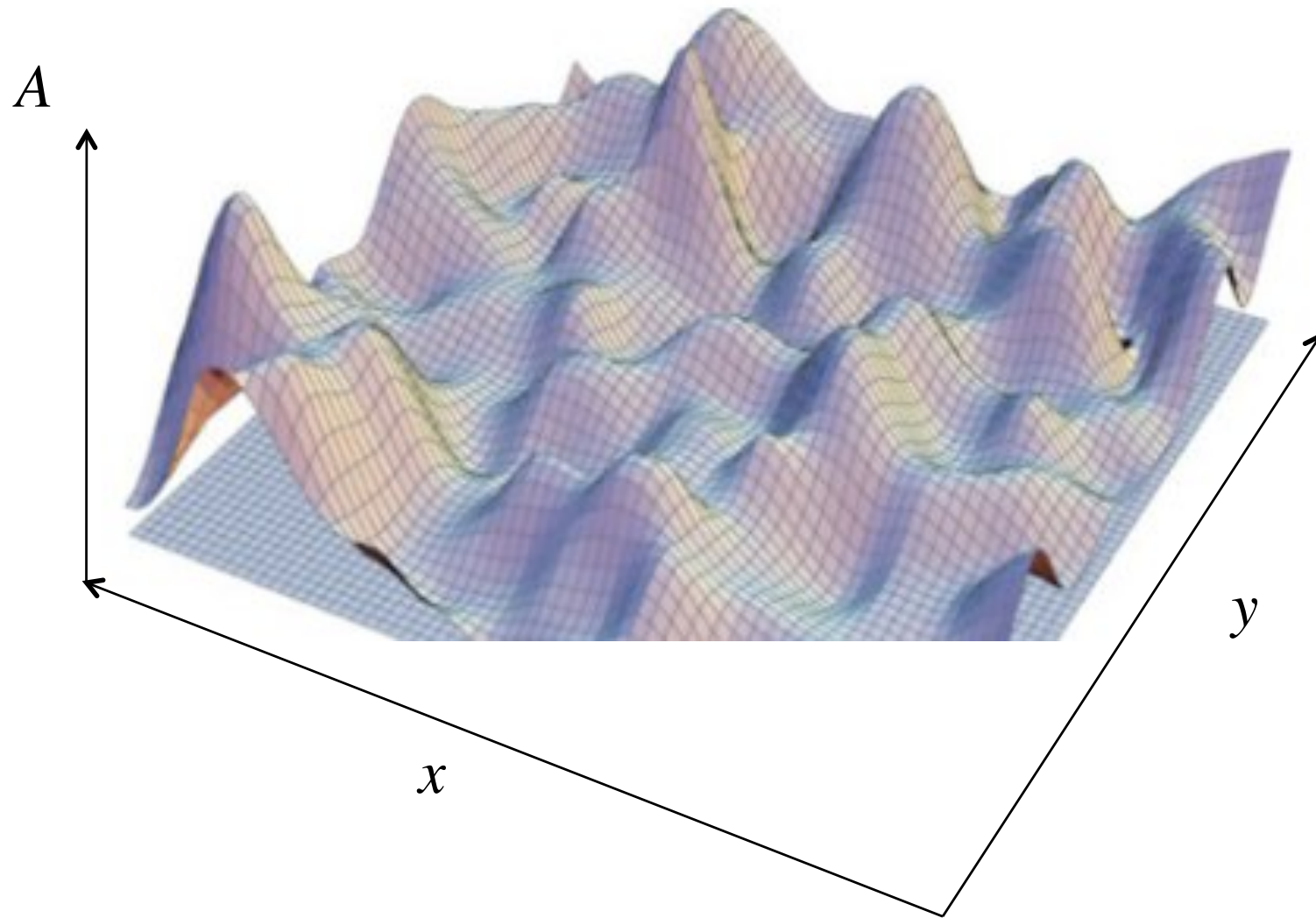
During Inflation



After Inflation



Ripples in Space



Stretched to cosmological distances

GRAVITATIONAL WAVE PERTURBATIONS

$$\delta S = \frac{1}{2} \int d^3x d\eta \frac{a^2}{2\kappa^2} \left[(h'_{ij})^2 - (\nabla h_{ij})^2 \right]$$

tensor field h_{ij} considered as a quantum field,

$$\hat{h}_{ij}(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \left[h_{\mathbf{k}}(\eta) e_{ij}(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda} e^{i\mathbf{k}\cdot\mathbf{x}} + h.c. \right],$$

$e_{ij}(\mathbf{k}, \lambda)$ are the two polarization tensors,

satisfying symmetric, transverse and traceless conditions

$$e_{ij} = e_{ji}, \quad k^i e_{ij} = 0, \quad e_{ii} = 0,$$

$$e_{ij}(-\mathbf{k}, \lambda) = e_{ij}^*(\mathbf{k}, \lambda), \quad \sum_{\lambda} e_{ij}^*(\mathbf{k}, \lambda) e^{ij}(\mathbf{k}, \lambda) = 4,$$

TENSOR MODE EQUATION

gauge invariant tensor amplitude

$$v_k(\eta) = \frac{a}{\sqrt{2\kappa}} h_k(\eta)$$

decoupled in linear perturbation theory,

$$v_k'' + \left(k^2 - \frac{a''}{a} \right) v_k = 0$$

For constant slow-roll parameters, the potential becomes

$$\frac{a''}{a} = 2\mathcal{H}^2 \left(1 - \frac{\epsilon}{2} \right) = \frac{1}{\eta^2} \left(\mu^2 - \frac{1}{4} \right),$$

$$\mu = \frac{1}{1 - \epsilon} + \frac{1}{2}.$$

EXACT SOLUTIONS

in the two asymptotic regimes,

$$v_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad k \gg aH \quad \text{Minkowsky}$$

$$v_k = C_3(k) a \quad k \ll aH \quad \text{superhorizon}$$

exact solutions

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\mu+\frac{1}{2})\frac{\pi}{2}} (-\eta)^{1/2} H_\mu^{(1)}(-k\eta)$$

$$\text{In the limit } k\eta \rightarrow 0, \quad |v_k| = \frac{C(\mu)}{\sqrt{2k}} \left(\frac{k}{aH}\right)^{\frac{1}{2}-\mu}$$

Since h_k becomes constant on superhorizon scales, evaluate the tensor metric perturbation when it reentered during the radiation or matter era directly in terms of its value during inflation.

Predictions of Inflation

BIG BANG

Inflation

Quantum
fluctuations

Radiation background
anisotropies

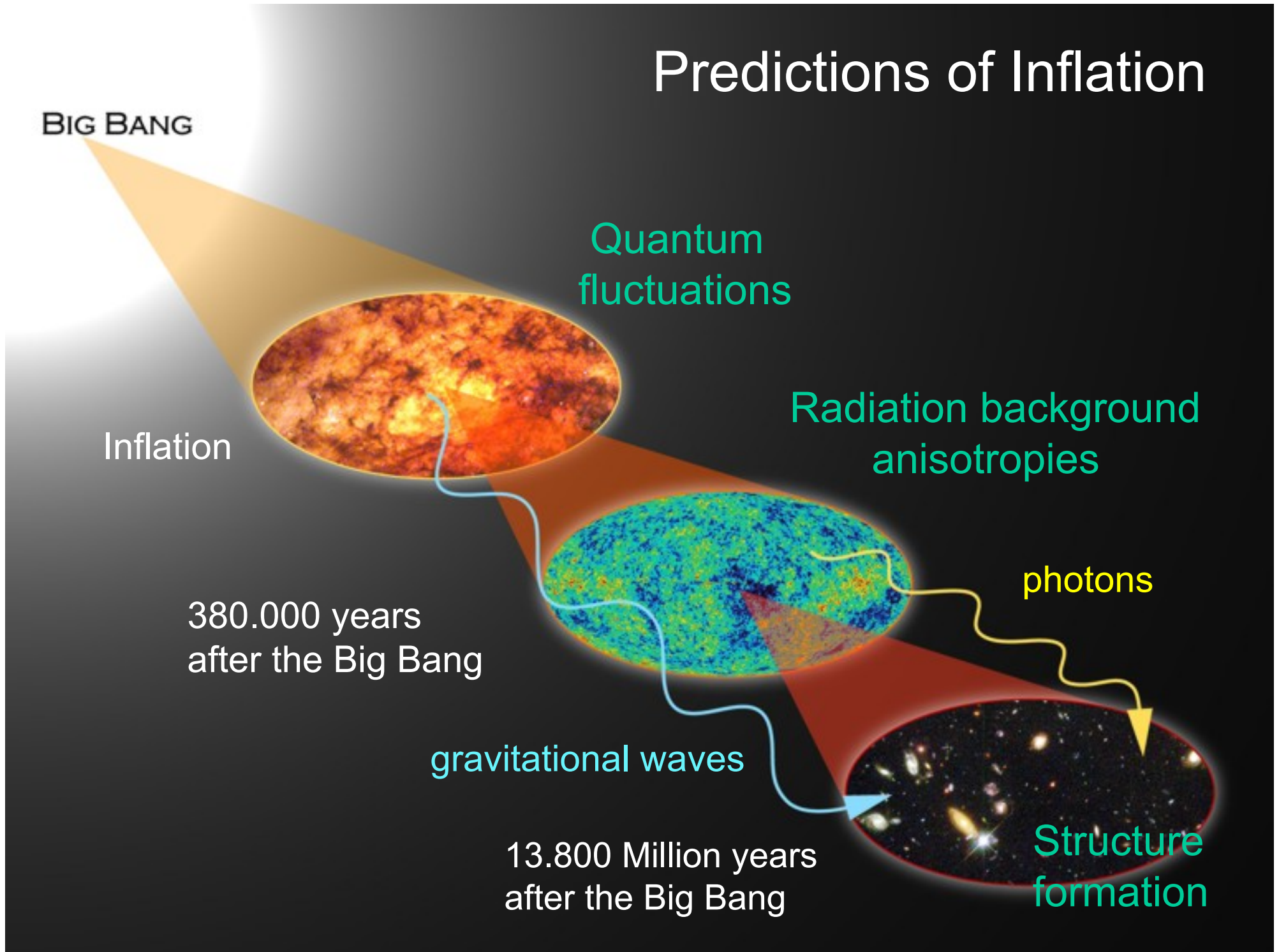
380.000 years
after the Big Bang

photons

gravitational waves

13.800 Million years
after the Big Bang

Structure
formation



Basic Inflationary Predictions

Geometry and matter:

- Homogeneity (acausal origin)
- Flat spatial sections (exp. growth)
- No appreciable topology (exp.growth)
- Origin matter & radiation (reheating)

Metric Perturbations:

- Gaussian spectrum (ground state)
- Aprox. scale invariant (slow roll cond.)
- Adiabatic density fluctuations (single fluid)
- Gravitational waves (tensor metric pert.)
- No vector perturbations (no defects)

Cosmological Observations

Cosmic Microwave Background:

- Temperature Anisotropies (WMAP9+Planck)
- Polarization Anisotropies (BICEP2+Planck)

Large Scale Structure:

- Matter Power Spectrum (2dFGRS+SDSS)
- Baryon Acoustic Oscillations (BOSS+DES)
- Weak Lensing (KIDS, DES, LSST, Euclid)





















